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The ladder problem: Painlevé integrability and explicit solution

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Abstract

We consider the n-dimensional ladder system, that is the homogeneous quadratic system of first-order differential equations of the form $\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij} x_j$, $i = 1, n$, where $(a_{ij}) = (i + 1 - j)$, $i, j = 1, n$, introduced by Imai and Hirata (2002 *Preprint* nlin.SI/0212007 v1 3). The ladder system is found to be integrable for all n in terms of the Painlevé analysis and its solution is explicitly given.

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1. Introduction

In two recent papers Imai and Hirata developed a necessary condition for the existence of Lie point symmetries in n-dimensional systems of first-order ordinary differential equations [3] and applied the ideas developed there to establish a new integrable family in the class of Lotka–Volterra systems [4].

At first sight the former result could raise questioning glances since 'everyone' knows that a system of first-order equations, be it linear or nonlinear, autonomous or nonautonomous, possesses an infinite number of Lie symmetries. However, as Imai and Hirata make clear in the text [3], they are concerned with autonomous symmetries of the form

$$\Gamma = \phi_i(x)\partial_{x_i} \tag{1.1}$$

for the autonomous system

$$\dot{\mathbf{x}} = \mathbf{g}(x) \tag{1.2}$$

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where $\mathbf{g}(x)$ is analytic in the neighbourhood of a fixed point. Moreover, the symmetry, that is to say the coefficient functions ϕ_i, is required to be analytic in the neighbourhood of some fixed point.

That, of course, makes for an entirely different story. These additional requirements mean that symmetries—incidentally only the point subset of symmetries is considered; nonlocal symmetries, the only other type for a system of first-order equations, are not considered—of some systems are rejected since the coefficient functions are not analytic. One advantage of this approach is that criteria are developed which enable the search for acceptable symmetries to be implemented in a finite procedure. This is in pleasant contrast to the general case wherein one is confronted with an infinite number of possibilities.

One may wonder a little at the strength of the restriction to a symmetry of the form (1.1), particularly in type of system presented by way of example in [3] and more generally considered in [4]. Autonomous homogeneous quadratic systems automatically possess the two Lie point symmetries

$$\Gamma_1 = \partial_t \quad \text{and} \quad \Gamma_2 = -t\partial_t + x_i\partial_{x_i}. \tag{1.3}$$

If one is looking towards the construction of first integrals for systems of first-order equations, one does not want symmetries with time-dependent coefficient functions. However, the combination of Γ_1 and Γ_2 effectively achieves that absence. In fact one can have much worse looking symmetries than Γ_2 for autonomous systems [6, 7].

In their application of the results of [3] to n-dimensional homogeneous, that is to say quadratic, Lotka–Volterra systems of the form

$$\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, n \tag{1.4}$$

which arise in a variety of applications, Imai and Hirata [4] considered two specific structures for the n-dimensional homogeneous Lotka–Volterra system, the ladder system and the generalized ladder system. The former is characterized by the possession of an $(n-1)$-dimensional Abelian Lie algebra of analytic symmetries of type (1.1). The constraints of the latter so that it is integrable in terms of analytic functions are determined by the imposition of an Abelian-like algebra. Imai and Hirata [4] observed that most studies of Lotka–Volterra systems and generalizations [1] are based on the determination of first integrals to establish integrability whereas they use an algebraic approach. Of course, the two approaches are not unrelated [2]. The constraints imposed on the symmetries they used must of necessity produce only first integrals as is evident from the approach using Jacobi's last multiplier as found in [10] and at most $n-1$ since there must be at least one invariant.

In this paper we further consider the properties of the ladder system with particular attention to the practical aspects of the integrability of the system in terms of the Painlevé analysis, explicit reduction and generation of solutions. This paper is structured as follows. Section 3 is devoted to the n-dimensional ladder system, which is proved to be integrable for all values of n in terms of the Painlevé analysis. We conclude in section 4 with some general remarks.

2. The ladder problem: general properties

The n-dimensional ladder system is the system (1.4) with the specific coefficient matrix

$$A = (a_{ij}) = \begin{bmatrix} 1 & 0 & \cdots & -n+2 \\ 2 & 1 & \cdots & -n+3 \\ \vdots & \vdots & \ddots & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix} \tag{2.1}$$

where

$$(a_{ij}) = i + 1 - j \tag{2.2}$$

$$a_{i+1,j} - a_{ij} = 1. \tag{2.3}$$

The ladder system has certain general properties.

Proposition 1.

$$\det A = \begin{cases} 1, & n = 1, 2 \\ 0, & n > 2. \end{cases}$$

Proof. For $n \geqslant 3$

$$\det A = \begin{vmatrix} 1 & 0 & -1 & -2 & \cdots & -n+2 \\ 2 & 1 & 0 & -1 & \cdots & -n+3 \\ 3 & 2 & 1 & 0 & \cdots & -n+4 \\ 4 & 3 & 2 & 1 & \cdots & -n+5 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 & -2 & \cdots & -n+2 \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 2 & 4 & 6 & \cdots & 2(n-1) \\ 4 & 3 & 2 & 1 & \cdots & -n+5 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix} = 0 \tag{2.4}$$

because the third row is a multiple of the second row, where we have made use of the row operations $R_2' = R_2 - 2R_1$ and $R_3' = R_3 - 3R_1$.

For $n = 1$, $\det A = \det 1 = 1$ and, for $n = 2$,

$$\det A = \det \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1. \qquad \square$$

Corollary. *The rank of the matrix A is two for $n > 2$.*

Proposition 2. *The n-dimensional ladder system can be considered as the simplest Riccati equation for $\sum_{i=1}^{n} x_i$.*

Proof. Consider the system of equations (1.4). After summation on $i = 1, n$ we obtain

$$\sum_{i=1}^{n} \dot{x}_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

$$\Leftrightarrow \quad \left(\sum_{i=1}^{n} x_i\right)^{\cdot} = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} (a_{ij} + a_{ji}) x_i x_j$$

$$\stackrel{\forall i,j}{\Leftrightarrow} \quad \left(\sum_{i=1}^{n} x_i\right)^{\cdot} = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} 2 x_i x_j$$

$$\Leftrightarrow \quad \left(\sum_{i=1}^{n} x_i\right)^{\cdot} = \left(\sum_{i=1}^{n} x_i\right)^2$$

with solution

$$\sum_{i=1}^{n} x_i = -\frac{1}{t - t_0}. \tag{2.5}$$

□

Proposition 3. *The ratio of two successive solutions of the ladder system is*

$$\frac{x_{i+1}}{x_i} = \frac{K_i}{t - t_0}.$$

Proof. For any $i = 1, n - 1$

$$\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij} x_j$$

$$\dot{x}_{i+1} = x_{i+1} \sum_{j=1}^{n} a_{i+1,j} x_j$$

whence

$$\begin{aligned} x_i \dot{x}_{i+1} - x_{i+1} \dot{x}_i &= x_i x_{i+1} \sum_{j=1}^{n} a_{i+1,j} x_j - x_{i+1} x_i \sum_{j=1}^{n} a_{ij} x_j \\ &= x_i x_{i+1} \left[\sum_{j=1}^{n} (a_{i+1,j} - a_{ij}) x_j \right] \\ &\stackrel{(2.3)}{=} x_i x_{i+1} \sum_{j=1}^{n} x_j \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \frac{x_i \dot{x}_{i+1} - x_{i+1} \dot{x}_i}{x_i^2} &= \frac{x_{i+1}}{x_i} \sum_{j=1}^{n} x_j \\ \Leftrightarrow \left(\frac{x_{i+1}}{x_i}\right)^{\cdot} &= \left(\frac{x_{i+1}}{x_i}\right) \sum_{j=1}^{n} x_j \\ \stackrel{(2.5)}{\Leftrightarrow} \frac{x_{i+1}}{x_i} &= \frac{K_i}{t - t_0} \end{aligned} \tag{2.6}$$

from proposition 2. □

3. Painlevé analysis

Proposition 4. *The resonances of the n-dimensional ladder system are* $-1, 0(n-2)$*-fold and 1 in the Painlevé analysis.*

Proof. To determine the leading order behaviour let

$$x_i = \alpha_i \tau^{p_i}. \tag{3.1}$$

When (3.1) is substituted into (1.4), we obtain

$$\alpha_i p_i \tau^{p_i - 1} = \alpha_i \tau^{p_i} \sum_{j=1}^{n} a_{ij} \alpha_j \tau^{p_j} \tag{3.2}$$

from which it is evident that $p_j = -1$, $j = 1, n$, i.e. the singular behaviour of the leading order terms is that of a simple pole. Equation (3.2) for $p_j = -1$, $j = 1, n$, takes the form

$$\sum_{j=1}^{n} a_{ij}\alpha_j = -1 \qquad i = 1, n$$
$$\Longrightarrow A\boldsymbol{\alpha} = \mathbf{c}$$

where $\mathbf{c}^T = (-1, -1, \ldots, -1)$.

Following the same procedure used in proposition 1 of section 1 and by considering the augmented matrix $[A, \mathbf{c}]$, we perform the following row operations

$$\begin{aligned} \mathrm{Row}_i^{(1)} &= \mathrm{Row}_i - i\,\mathrm{Row}_1 \qquad && i = 2, n \\ \mathrm{Row}_i^{(2)} &= \mathrm{Row}_i^{(1)} - (i-1)\mathrm{Row}_2^{(1)} \qquad && i = 3, n \end{aligned} \tag{3.3}$$

and the system reduces to

$$\alpha_1 = -1 + \alpha_3 + 2\alpha_4 + \cdots = -1 + \sum_{j=3}^{n} (j-2)\alpha_j \tag{3.4}$$

$$\alpha_2 = 1 - 2\alpha_3 - 3\alpha_4 - \cdots = 1 - \sum_{j=3}^{n} (j-1)\alpha_j. \tag{3.5}$$

In order to determine the resonances we set

$$x_i = \alpha_i \tau^{-1} + \mu_i \tau^{r-1}$$

with which, when we substitute into (1.4) and take all terms linear in the μ_i, we obtain

$$\begin{aligned} (r-1)\mu_i &= \alpha_i \sum_{j=1}^{n} a_{ij}\mu_j + \mu_i \sum_{j=1}^{n} a_{ij}\alpha_j \\ &\Leftrightarrow (r-1)\mu_i - \mu_i \sum_{j=1}^{n} a_{ij}\alpha_j - \alpha_i a_{ii}\mu_i - \alpha_i \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\mu_j = 0. \end{aligned} \tag{3.6}$$

Separating by coefficients of μ_i and not μ_i in equation (3.6) and by using (2.2) we obtain

$$\begin{aligned}
r - 1 - \sum_{j=1}^{n}(i+1-j)\alpha_j - \alpha_i
&= r - 1 - i\alpha_1 - (i-1)\alpha_2 - \sum_{j=3}^{n}(i+1-j)\alpha_j - \alpha_i \\
&\overset{(3.4)}{\underset{(3.5)}{=}} r - 1 - i\left[-1 + \sum_{j=3}^{n}(j-2)\alpha_j\right] - (i-1)\left[1 - \sum_{j=3}^{n}(j-1)\alpha_j\right] \\
&\quad - \sum_{j=3}^{n}(i+1-j)\alpha_j - \alpha_i \\
&= r - \alpha_i
\end{aligned}$$

and

$$-\alpha_i a_{ij} = -\alpha_i(i+1-j)$$

respectively.

Thus, we create the system

$$M\boldsymbol{\mu} = 0 \tag{3.7}$$

where the matrix M is defined as

$$M = \begin{bmatrix}
r-\alpha_1 & 0 & \alpha_1 & 2\alpha_1 & 3\alpha_1 & \cdots & (n-2)\alpha_1 \\
-2\alpha_2 & r-\alpha_2 & 0 & \alpha_2 & 2\alpha_2 & \cdots & (n-3)\alpha_2 \\
\vdots & \vdots & & & & & \vdots \\
-i\alpha_i & -(i-1)\alpha_i & \cdots & r-\alpha_i & 0 & \cdots & (n-i-1)\alpha_i \\
\vdots & & & & & \vdots & \\
-n\alpha_n & -(n-1)\alpha_n & \cdots & \cdots & \cdots & -2\alpha_n & r-\alpha_n
\end{bmatrix}. \tag{3.8}$$

We observe that M can equally take the more compact form $M = rI - N$, where $N_{ij} = (i+1-j)\alpha_i$. Since, for a fixed j $(= 1, n)$,

$$\begin{aligned}
\sum_{i=1}^{n} N_{ij} = \sum_{i=1}^{n}(i+1-j)\alpha_i
&= (2-j)\alpha_1 + (3-j)\alpha_2 + \sum_{i=3}^{n}(i+1-j)\alpha_i \\
&\overset{(3.4)}{\underset{(3.5)}{=}} (2-j)\left[-1 + \sum_{i=3}^{n}(i-2)\alpha_i\right] + (3-j)\left[1 - \sum_{i=3}^{n}(i-1)\alpha_i\right] \\
&\quad + \sum_{i=3}^{n}(i+1-j)\alpha_i \\
&= 1
\end{aligned}$$

by adding all rows for every $j = 1, n$ we obtain

$$|M|_{n\times n} = (r-1)\begin{vmatrix} & (rI-N)_{n-1,n} & \\ 1 \quad 1 & 1\ldots & 1 \end{vmatrix} = (r-1)|rI - P| \tag{3.9}$$

where $P_{ij} = \alpha_i(n-j)$. The elements of the $(n-1)\times(n-1)$ matrix P are constructed by the subtraction of the nth column of the middle term in (3.9) from the other columns.

The minors of order greater than or equal to 2 of P are zero since

$$\begin{vmatrix} \alpha_i(n-j) & \alpha_i(n-j-l) \\ \alpha_{i+k}(n-j) & \alpha_{i+k}(n-j-l) \end{vmatrix} = \alpha_i\alpha_{i+1}(n-j)(n-j-l)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

for all $k, l = 2, n$. We use the Laplace expansion of a determinant to evaluate $|rI - P|$ and (3.9) yields

$$|M| = (r-1)(r^{n-1} - \mathrm{Tr}(P)r^{n-2})$$

where $\mathrm{Tr}(P)$ stands for the trace of P given by

$$\begin{aligned}\mathrm{Tr}(P) &= \sum_{i=1}^{n-1}\alpha_i(n-i) \\ &= (n-1)\alpha_1 + (n-2)\alpha_2 + \sum_{i=3}^{n-1}\alpha_i(n-i) \\ &\overset{(3.4)}{\underset{(3.5)}{=}} (n-1)\left[-1 + \sum_{i=3}^{n}(i-2)\alpha_i\right] + (n-2)\left[-1 - \sum_{i=3}^{n}(i-1)\alpha_i\right] \\ &\quad + \sum_{i=3}^{n-1}\alpha_i(n-i) \\ &= -1.\end{aligned}$$

Thus

$$|M| = (r-1)(r^{n-1} + r^{n-2}) = (r-1)r^{n-2}(r+1) \tag{3.10}$$

and $|M| = 0 \Longrightarrow r = \pm 1, 0(n-2)$. □

Corollary 1. *The n-dimension ladder system is consistent at all resonances.*

Corollary 2. *The n-dimensional ladder system passes the Painlevé test.*

4. Remarks and observations

In the above we have shown that the n-dimensional ladder system introduced by Imai and Hirata [4] passes the Painlevé test for all n. In propositions 2 and 3 we provide the results necessary to obtain the general solution of the n-dimensional ladder system. Recall that in proposition 2 we showed that

$$\sum_{i=1}^{n} x_i = -\frac{1}{t-t_0} \tag{4.1}$$

and in proposition 3 that

$$\frac{x_{i+1}}{x_i} = \frac{K_i}{t-t_0} \tag{4.2}$$

$$\Longrightarrow \frac{x_{i+j}}{x_i} = \frac{\prod_{k=1}^{j} K_{i+k}}{(t-t_0)^j}. \tag{4.3}$$

When we use (4.3), (4.1) becomes

$$x_1\left(1+\sum_{i=1}^{n-1}\frac{\prod_{k=1}^{i}K_k}{(t-t_0)^i}\right)=-\frac{1}{t-t_0} \tag{4.4}$$

whence

$$x_1=-\frac{(t-t_0)^{n-2}}{\sum_{j=0}^{n-1}\left(\prod_{k=0}^{j}K_k\right)(t-t_0)^{n-1-j}}. \tag{4.5}$$

Again when we use (4.3), it follows that

$$x_{i+1}=x_1\frac{\prod_{j=1}^{i}K_j}{(t-t_0)^i}. \tag{4.6}$$

In the explicit statement of the solution to the ladder problem given in (4.5) and (4.6) we see that the ladder system possesses the Painlevé property since the only singularities are movable poles. This completes the demonstration of the integrability of the system already suggested by its passing of the Painlevé test as shown in proposition 4 and its corollaries.

Imai and Hirata [4] introduce the ladder system by definition. In proposition 2 we showed that the sum of the equations of the system gives the elementary Riccati equation

$$\dot{x}=x^2 \tag{4.7}$$

where

$$x=x_1+x_2+\cdots+x_n. \tag{4.8}$$

We may view this process in reverse. Starting from the elementary Riccati equation, (4.7), we decompose the dependent variable into n-dependent variables as in (4.8) and then separate the single equation into a system of n equations, one for each x_i, $i=1,n$, in such a way that the sum of the right-hand sides of the component equations gives $(x_1+x_2+\cdots+x_n)^2$. The ladder system of Imai and Hirata has precisely this property.

It is interesting that this method of decomposition preserves the well-known Painlevé property of the Riccati equation in the constituent equations of the ladder system. It is not invariable that decomposition preserves the Painlevé property. Leach *et al* [9] presented an instance, in which this preservation was not observed. The example was of a four-dimensional quadratic system which is the decomposition of a third-order rather than the expected fourth-order scalar equation and so has an element of the decomposition described above. The conditions under which a process of decomposition preserves the Painlevé property remain to be delineated.

The low-dimensional members of the ladder system are related to equations well known in physics. In the case that $n=1$ we have the Riccati equation. In the case that $n=2$ the two-dimensional system is reducible to the scalar second-order Ermakov–Pinney equation. The three-dimensional system is reducible to the scalar third-order equation of maximal symmetry. When $n=4$ the reduction to a scalar higher-order equation is not possible. However, the system can be reduced to two second-order equations. This is reflected in the occurrence of two arbitrary constants in the coefficients of the leading order terms. For ladder systems of higher dimension the number of equations can only be reduced by two because of the arbitrariness of $(n-2)$ of the coefficients of the leading order terms as determined by the Painlevé analysis.

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References

[1] Laurent C and Jaume L 1999 Integrability and algebraic solutions for the 2-D Lotka–Volterra system *Dynamical Systems, Plasmas and Gravitation* ed P G L Leach, S É Bouquet, J-L Rouet and E Fijkalow (Heidelberg: Springer) pp 243–52

[2] Flessas G P, Govinder K S and Leach P G L 1997 Characterisation of the algebraic properties of first integrals of scalar ordinary differential equations of maximal symmetry *J. Math. Anal. Appl.* **212** 349–74

[3] Yoshihiro H and Kenji I 2002 A necessary condition for existence of Lie symmetries in general systems of ordinary differential equations *J. Phys. Soc. Japan* **71** 2396–400

[4] Kenji I and Yoshihiro H 2002 New integrable family in the *n*-dimensional homogeneous Lotka–Volterra systems with abelian Lie algebra *Preprint* nlin.SI/0212007 v1

[5] Ince E L 1927 *Ordinary Differential Equations* (London: Longmans Green)

[6] Krause J 1994 On the complete symmetry group of the classical Kepler system *J. Math. Phys.* **35** 5734–48

[7] Krause J 1995 On the complete symmetry group of the Kepler problem *Proc. 22nd International Colloquium on Group Theoretical Methods in Physics* ed A Arima (Singapore: World Scientific) pp 286–90

[8] Leach P G L, Govinder K S and Abraham-Shrauner B 1999 Symmetries of first integrals and their associated differential equations *J. Math. Anal. Appl.* **235** 58–83

[9] Leach P G L, Cotsakis S and Flessas G P 2000 Symmetry, singularity and integrability in complex dynamics: I The reduction problem *J. Nonlinear Math. Phys.* **7** 445–79

[10] Nucci M C and Leach P G L 2002 Jacobi's last multiplier and the complete symmetry group of the Euler-Poinsot system *J. Nonlinear Math. Phys.* **9** (suppl 2) 110–21